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MINIMUM IMBEDDINGS OF COMPACT SYMMETRIC SPACES OF RANK ONE

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1. Introduction

Let M be a compact differentiable manifold of dimension n, and

$$\varphi : M \to R^{n+k} \tag{1.1}$$

an immersion of M into a Euclidean space R^{n+k} of dimension $n+k$. The total curvature, in the sense of Chern and Lashof [1], [12], can be defined as follows.

Let B be the set of unit normal vectors of M in R^{n+k}. Then B is a bundle of $(k-1)$-sphere over M and is a manifold of dimension $n+k-1$. Let S be the unit $(n+k-1)$-sphere of R^{n+k}, $d\sigma$ the volume element of S, and

$$c_{n+k-1} = \int_S d\sigma \tag{1.2}$$

the volume of S. If

$$\nu : B \to S \tag{1.3}$$

is the Gauss map, which assigns each unit normal vector of B the unit vector through the origin and parallel to the normal vector, then the total curvature of the immersed manifold M is defined as

$$\frac{1}{c_{n+k-1}} \int_B |\nu^* d\sigma| \,. \tag{1.4}$$

Since the total curvature depends on M as well as $\varphi : M \to R^{n+k}$, we shall denote it by $\tau(M, \varphi, R^{n+k})$ or simply by $\tau(\varphi)$.

The height function h_a in the direction $a \in R^{n+k}$ takes the value

$$h_a(x) = (a, \varphi(x)) \tag{1.5}$$

at $x \in M$, where (,) denotes the usual inner product on R^{n+k}. $x \in M$ is a

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critical point of h_a, $a \in \boldsymbol{R}^{n+k}$, if and only if a is normal to M, and h_a, $a \in S$ has a degenerate critical point if and only if a is a critical value of the map $\nu: B \to S$. By Sard's theorem, the image of the set of critical points of ν has measure 0 in S. Hence for almost all $a \in S$, h_a has only nondegenerate critical points. Let $\beta(M, f)$ denote the number of critical points of a differentiable function f defined over M. Then $\beta(M, h_a)$ is well defined and is finite for almost all $a \in S$. We have [12]

$$\tau(M, \varphi, \boldsymbol{R}^{n+k}) = \int_{a \in S} \beta(M, h_a) d\sigma \,. \tag{1.6}$$

So, to evaluate the total curvature $\tau(\varphi)$, it is sufficient to determine the number of critical points of the height functions.

Let F be the set of differentiable functions on M whose critical points are all nondegenerate, and define

$$\beta(M) = \inf_{f \in F} \beta(M, f) \,. \tag{1.7}$$

It follows from Morse inequality [13, p. 29] that

$$\beta(M) \geq b(M) = \sum b_i(M) \,, \tag{1.8}$$

where $b_i(M)$ is the i-th Betti number and $b(M)$ the sum of Betti numbers of M. Kuiper [12] has shown that

$$\inf_{\varphi, k} \tau(M, \varphi, \boldsymbol{R}^{n+k}) = \beta(M) \,. \tag{1.9}$$

An immersion $\varphi: M \to \boldsymbol{R}^{n+k}$ is said to be *minimal* if $\tau(\varphi) = \beta(M)$. Given a compact differentiable manifold M, it is not true in general that M can always be minimally immersed. As Kuiper has pointed out [12], if M is an exotic sphere, it admits a function with two critical points and hence $\beta(M) = 2$. On the other hand, by a theorem of Chern and Lashof [1], an immersed compact differentiable manifold M with $\tau(M, \varphi, \boldsymbol{R}^{n+k}) = 2$ is a convex hypersurface in some $\boldsymbol{R}^{n+1} \subset \boldsymbol{R}^{n+k}$, which implies that M is diffeomorphic to an ordinary sphere. Ferus [3] proved that every imbedding of an exotic n-sphere $(n \geq 5)$ in $\boldsymbol{R}^{n+2}$ has a total curvature ≥ 4.

If $\varphi(M)$ is not contained in any hyperplane of $\boldsymbol{R}^{n+k}$, then we say that the immersion $\varphi: M \to \boldsymbol{R}^{n+k}$ is *substantial*. A theorem of Kuiper [12] asserts that if $\varphi: M \to \boldsymbol{R}^{n+k}$ is minimal and substantial, then $k \leq n(n+1)/2$. He also gives examples of minimal and substantial imbeddings of various codimensions k, $1 \leq k \leq n(n+1)/2$, [12, pp. 82-83]. In particular, the Hopf imbeddings of real projective space $P_n(\boldsymbol{R})$ into $\boldsymbol{R}^{n+k}$, $n+1 \leq k \leq n(n+1)/2$, are minimal and substantial. In the same paper [12, p. 86], he exhibited a minimum imbedding of the real projective plane $P_2(\boldsymbol{R})$ into $\boldsymbol{R}^4$.

Kobayashi [11] proved that every compact homogenous Kähler manifold can be minimally imbedded into a Euclidean space. In particular, the Mannoury imbedding [7, pp. 150–151] of a complex projective space $P_n(C)$ into $R^{(n+1)^2-1}$ is shown to be minimal (cf. Remark 2.6).

In this paper, we are going to construct minimum imbeddings of compact symmetric spaces of rank one in a unified fashion. Beside being minimal and substantial, these imbeddings are also equivariant and isometric.

The problem is trivial for spheres. We will treat real, complex and quaternionic projective spaces in § 2, and the Cayley projective plane in § 3.

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2. Projective spaces

Throughout this section, F will denote the field R of real numbers, the field C of complex numbers or the field Q of quaternions. In a natural way, $R \subset C \subset Q$. For each element x of F, we define the conjugate of x as follows. If

$$x = x_0 + x_1 i + x_2 j + x_3 k \in Q , \tag{2.1}$$

with $x_0, x_1, x_2, x_3 \in R$, then

$$\bar{x} = x_0 - x_1 i - x_2 j - x_3 k . \tag{2.2}$$

If x is in C, then $\bar{x}$ coincides with the ordinary complex conjugate of x. If x is in R, then $\bar{x} = x$.

It is convenient for us to define

$$d = d(F) = \begin{cases} 1 & \text{if} \quad F = R , \\ 2 & \text{if} \quad F = C , \\ 4 & \text{if} \quad F = Q . \end{cases} \tag{2.3}$$

Let $x = (x_0, \cdots, x_n) \in F^{n+1}$. A matrix $A = (a_{ij})$, $0 \leq i, j \leq n$, operates on F^{n+1} by the rule:

$$Ax = \begin{pmatrix} a_{00} & \cdots & a_{0n} \\ \cdots & \cdots & \cdots \\ a_{n0} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix} . \tag{2.4}$$

The transpose and conjugate of a matrix A are denoted by ${}^tA$ and $\bar{A}$, respectively; A^*dentes ${}^t\bar{A}$. We will use the following notations:

(2.5) $M(n+1, F) =$ the space of all $(n+1) \times (n+1)$ matrices over F.

(2.6) $H(n+1, F) = \{A \in M(n+1, F) \mid A^* = A\}$, the space of all $(n+1) \times (n+1)$ Hermitian matrices over F.

If $A \in \mathrm{H}(n+1, R)$, then A is symmetric.

$$U(n+1, F) = \{X \in M(n+1, F) \mid X^*X = I\}\ , \tag{2.7}$$

where I denotes the identity matrix. Then $U(n+1, R) = 0(n+1)$, $U(n+1, C) = U(n+1)$ and $U(n+1, Q) = Sp(n+1)$, in standard notations.

F^{n+1} can be considered as a Euclidean space of dimension $(n+1)d$. The usual inner product for $F^{n+1} = R^{(n+1)d}$ is defined as

$$(x, y) = Re(x^*y)\ , \tag{2.8}$$

where x and $y \in F^{n+1}$ are represented as column matrices. $M(n+1, F)$ can also be considered as a Euclidean space of dimension $(n+1)^2d$, and

$$(A, B) = ReTr(AB^*)\ , \qquad A, B \in M(n+1, F)\ , \tag{2.9}$$

defines the usual inner product. If A and B belong to $H(n+1, F)$, then

$$(A, B) = Tr(AB)\ . \tag{2.10}$$

We will endow $H(n+1, F)$ with this induced inner product.

Let $P_n(F)$ denote the projective space over F. Consider $P_n(F)$ as the quotient space of unit $((n+1)d-1)$-sphere $\{x = (x_0, \cdots, x_n) \in F^{n+1} \mid x^*x = 1\}$ obtained by identifying $(x_0, \cdots, x_n)$ with $(x_0\lambda, \cdots, x_n\lambda)$, where $\lambda \in F$ and $|\lambda| = 1$. Hence for $x \in P_n(F)$, we can use homogeous coordinates

$$x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}, \quad \text{with } x^*x = 1\ . \tag{2.11}$$

Consider the following map

$$\varphi : P_n(F) \to H(n+1, F) \tag{2.12}$$

such that

$$\varphi(x) = xx^* = \begin{pmatrix} |x_0|^2 & x_0\bar{x}_1 & \cdots & x_0\bar{x}_n \\ x_1\bar{x}_0 & |x_1|^2 & \cdots & x_1\bar{x}_n \\ \cdots & \cdots & \cdots & \cdots \\ x_n\bar{x}_0 & x_n\bar{x}_1 & \cdots & |x_n|^2 \end{pmatrix}. \tag{2.13}$$

It is clear that this is a well defined function from $P_n(F)$ into a Euclidean space of dimension $\binom{n+1}{2}d+n+1$. The conditions $x^*x=1$, $Tr(\varphi(x))=1$ and $\sum_{i=0}^{n}|x_i|^2=1$ are obviously equivalent to each other. It follows that the image of $P_n(F)$ under φ lies on the hyperplane

$$H_1(n+1,F)=\{X=(x_{ij})\in H(n+1,F)\,|\,\sum_{i=0}^{n}x_{ii}=1\}\,. \tag{2.14}$$

For x and $y\in P_n(F)$, $\varphi(x)=\varphi(y)$, which is equivalent to $xx^*=yy^*$, implies $x=y\lambda$, with $\lambda\in F$ and $|\lambda|=1$. Thus φ is a substantial imbedding of $P_n(F)$ into R^N, $N=\binom{n+1}{2}d+n$. We wish to show that φ is a minimum imbedding.

Let $U(n+1,F)$ act linearly on $M(n+1,F)$ in the obvious manner:

$$X(A)=XAX^*\,, \tag{2.15}$$

$X\in U(n+1,F)$ and $A\in M(n+1,F)$.

Lemma 2.1. *The action of $U(n+1,F)$ preserves inner product of $M(n+1,F)$.*

The proof is straightforward.

Lemm 2.2. *The imbedding*

$$\varphi:P_n(F)\to H(n+1,F)$$

is equivariant with respect to and invariant under the action of $U(n+1,F)$, i.e.,

$$\varphi(Xx)=X(\varphi(x))\in\varphi(P_n(F)) \tag{2.16}$$

for $x\in P_n(F)$ and $X\in U(n+1,F)$.

The proof is straightforward.

Let $A\in H(n+1,F)$ and h_A be the height function defined over $P_n(F)$ in the diretion A. Then

$$h_A(x)=(A,\varphi(x))=Tr(A\varphi(x))=Tr(A(xx^*)) \tag{2.17}$$

at $x\in P_n(F)$. By Lemmas 2.1 and 2.2,

$$h_{X(A)}(Xx)=h_A(x)\,,\quad X\in U(n+1,F)\,. \tag{2.18}$$

On the other hand, for each $A\in H(n+1,F)$, there exists an $X\in U(n+1,F)$ such that $X(A)=XAX^*$ is a diagonal matrix. (The fact is well known for R and C. We will deal with the quaternionic case in the appendix.)

Therefore, to study the critical points of h_A, we may assume that A is a diagonal matrix,

$$A = \begin{pmatrix} \lambda_0 & & & 0 \\ & \lambda_1 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} . \tag{2.19}$$

Then the height function takes the simple form:

$$h_A(x) = \sum_{i=0}^{n} \lambda_i |x_i|^2 , \quad x \in P_n(F) . \tag{2.20}$$

The following is a standard trick to determine the critical points of h_A on $P_n(F)$ [13, pp. 26–27].

Consider the following coordinate system. Let U_0 be the set of $x = (x_0, x_1, \cdots, x_n)$ with $x_0 \neq 0$, and let

$$\begin{gathered} |x_0| x_i x_0^{-1} = u_i , \\ u_i = u_{i0} + u_{i1} i + u_{i2} j + u_{i3} k \in F . \end{gathered} \tag{2.21}$$

Then

$$u_{i\alpha} : U_0 \to R , \quad 1 \leq i \leq n , \quad 0 \leq \alpha \leq d - 1 , \tag{2.22}$$

are the required coordinate functions mapping U_0 diffeomorphically onto the open unit ball in R^{nd}. Clearly

$$|x_i|^2 = \sum_{\alpha} u_{i\alpha}^2 , \qquad 0 \leq \alpha \leq d - 1 . \tag{2.23}$$

$$\begin{gathered} x_0^2 = 1 - \sum_{i=1}^{n} |x_i|^2 = 1 - \sum_{i,\alpha} u_{i\alpha}^2 , \\ 1 \leq i \leq n , \; 0 \leq \alpha \leq d - 1 , \end{gathered} \tag{2.24}$$

so that

$$\begin{gathered} h_A = \lambda_0 + \sum_{i,\alpha} (\lambda_i - \lambda_0) u_{i\alpha}^2 , \\ 1 \leq i \leq n , \; 0 \leq \alpha \leq d - 1 , \end{gathered} \tag{2.25}$$

throughout the coordinate neighborhood U_0. Thus the only critical point of h_A within U_0 lies at the center point

$$P_0 = (1, 0, \cdots, 0) \tag{2.26}$$

of the coordinate system. At this point, h_A is nondegenerate if and only if all other eigenvalues are distinct from λ_0.

Similarly one can consider other coordinate neighborhoods centered at the points

$$(2.27) \qquad P_1 = (0, 1, \cdots, 0), \cdots, P_n = (0, \cdots, 0, 1) ,$$

It follows that $P_0, P_1, \cdots, P_n$ are the only critical points of h_A. Thus we have

Theorem 2.3. *For* $A \in H(n+1, F)$ *the height function* h_A *defined over* $P_n(F)$ *is nondegenerate and has exactly* $n+1$ *isolated critical points if and only if all eigenvalues are distinct from each each other.*

Remark 2.4 (cf. [13]). Every nondegenerate height function has indices id, $0 \leq i \leq n$, respectively at $n+1$ different critical points. From the cell decomposition of such a function, it follows immediately that the sum of Betti numbers $b(P_n(F)) = n+1$ if $d(F) = 2$ or 4. But it is well known that $b(P_n(R)) = n+1$. Therefore by Morse inequality (1.8), every nondegenerate height function has $\beta(M)$ $(= n+1)$ critical points, and we have proved

Theorem 2.5. *The imbedding* (cf. (2.14))

$$\varphi : P_n(F) \rightarrow H_1(\mathrm{n} + 1, F)$$

is substantial, minimal, isometric and equivariant.

The assertion that φ is an isometric imbedding follows from the fact that there is a Riemannian metric, unique up to a constant factor, on $P_n(F)$ which is invariant under $U(n+1, F)$, and the fact that the metric on $P_n(F)$ induced by φ is invariant under $U(n+1, F)$ (Lemma 2.1). Or more generally, every equivariant imbedding of an irreducible symmetric space is isometric, since an invariant metric on a homogeneous space with irreducible linear isotropy group is unique up to a constant factor.

Remark 2.6. In our notations, the Mannoury imbedding of complex projective space $P_n(C)$ can be described as follows [7, pp. 150–151]:

Let $R^{(n+1)^2}$ be a Euclidean space with coordinate system (X^h, X^{hk}, Y^{hk}), where $h, k = 0, 1, \cdots, n$ and $h \neq k$. The imbedding $P_n(C) \rightarrow R^{(n+1)^2}$ is defined by

$$(2.28) \qquad \begin{aligned} X_h &= \sqrt{2}\, x_h x_k = \sqrt{2}\, |x_h|^2 , \\ X^{hk} &= x_h \bar{x}_k + \bar{x}_h x_k = 2\,\mathrm{Re}\, x_h \bar{x}_k , \\ Y^{hk} &= i(x_h \bar{x}_k - \bar{x}_h x_k) = 2\,\mathrm{Im}\, x_h \bar{x}_k . \end{aligned}$$

Then $P_n(C)$ lies in the hyperplane

$$(2.29) \qquad X^0 + \cdots + X^n = \sqrt{2}$$

of $R^{(n+1)^2}$. It is easy to see that this imbedding differs from ours only by an affine transformation. It follows from Theorem 2.5 that the Mannoury imbedding is minimal by the following theorem of Kuiper [12]:

Theorem 2.7. *If* $\varphi : M \to R^{n+k}$ *is minimal and* $A : R^{n+k} \to R^{n+k}$ *is an affine transformation, then* $A \circ \varphi$ *is minimal.*

Remark 2.8. In his paper "On isometric imbeddings of compact symmetric spaces" (to appear), Kobayashi exhibits the same type of imbeddings for a class of symmetric spaces and conjecture that they are all minimal.

Added in proof. Kobayashi and Takeuchi have recently proved the above conjecture.

3. Cayley projective plane

Let $x = x_0 + x_1 j_1 + \cdots + x_7 j_7$ be an element of the Cayley algebra over the real field. Denote

$$\bar{x} = x_0 - x_1 j_1 - \cdots - x_7 j_7 , \tag{3.1}$$

the conjugate of x. Then the norm $n(x)$ of x is equal to

$$x\bar{x} = x_0^2 + \cdots + x_7^2 , \tag{3.2}$$

and we have

$$n(xy) = n(x)n(y) , \qquad x, y \in \boldsymbol{Cay} . \tag{3.3}$$

An element $x \neq 0 \in \boldsymbol{Cay}$ has an inverse $\bar{x}/n(x)$.

If $H(3, \boldsymbol{Cay})$ is the space of 3×3 Hermitian Cayley matrices, then $H(3, \boldsymbol{Cay})$ is a Jordan algebra under the following multiplication [9]:

$$X \circ Y = \tfrac{1}{2}(XY + YX) , \qquad X, Y \in H(3, \boldsymbol{Cay}) . \tag{3.4}$$

For simplicity, we express an element $X \in H(3, \boldsymbol{Cay})$ in the form:

$$X = \begin{pmatrix} \xi_1 & x & \bar{z} \\ \bar{x} & \xi_2 & y \\ z & \bar{y} & \xi_3 \end{pmatrix} . \tag{3.5}$$

Using the usual matrix unit E_{ij} and setting $E_{ii} = E_i$, and let

$$x_{ij} = xE_{ij} + \bar{x}E_{ij} , \tag{3.6}$$

we can write

$$X = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3 + x_{12} + y_{23} + z_{31} . \tag{3.7}$$

The E_i are orthogonal idempotents, and the trace and norm of X are defined respectively as in [8]:

$$(3.8)\qquad Tr(X) = \xi_1 + \xi_2 + \xi_3 ,$$

$$(3.9)\qquad N(X) = \xi_1\xi_2\xi_3 + Tr((xy)z) - \xi_1 n(y) - \xi_2 n(z) - \xi_3 n(x) .$$

The minimum polynomial of X is defined as

$$(3.10)\quad N(\lambda I - X) = \lambda^3 - Tr(X)\lambda^2 + \tfrac{1}{2}(Tr(X)^2 - Tr(X^2))\lambda - N(X) ,$$

and we have

$$(3.11)\qquad X^3 - Tr(X)X^2 + \tfrac{1}{2}(Tr(X)^2 - Tr(X^2))X - N(X)I = 0 ,$$

where $I = E_1 + E_2 + E_3$ is the identity matrix.

Following Jordan [10], we can define the Cayley projective plane $P_2(\boldsymbol{Cay})$ as the set

$$(3.12)\qquad \{xx^* \in H(3, \boldsymbol{Cay}) \mid x^*x = 1, x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \boldsymbol{Cay}^3\} .$$

This set is equivalent to ([4], [6])

$$(3.13)\qquad \{X \in H(3, \boldsymbol{Cay}) \mid X \circ X = X, Tr(X) = 1\} ,$$

and is contained in

$$(3.14)\qquad H_1(3, \boldsymbol{Cay}) = \{X \in H(3, \boldsymbol{Cay}) \mid Tr(X) = 1\} .$$

Consider $H(3, \boldsymbol{Cay})$ as a Euclidean space of dimension 27 endowed with the inner product

$$(3.15)\qquad (X, Y) = Tr(X \circ Y) , \qquad X, Y \in H(3, \boldsymbol{Cay}) ,$$

which is induced from the usual inner product of $M(3, \boldsymbol{Cay})$, the space of 3×3 Cayley matrices considered as $\boldsymbol{R}^{72}$.

Lemma 3.1. *An automorphism of the Jordan algebra $H(3, \boldsymbol{Cay})$ preserves the inner product.*

Proof. From (3.11), the trace function is invariant under the automorphisms of $H(3, \boldsymbol{Cay})$. The rest is straightforward.

The following two results can be found in Freudenthal [4]:

Lemma 3.2 [4, p. 25]. *The automorphism group of $H(3, \boldsymbol{Cay})$ is the exceptional Lie group F_4.*

Lemma 3.3 [4, p. 26]. *For each $X \in H(3, \boldsymbol{Cay})$, there is an $a \in F_4$ such that*

$$(3.16)\qquad a(X) = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3$$

i.e. the elements of $H(3, \mathbf{Cay})$ can be diagonalized by the action of F_4.

Lemma 3.4. *$P_2(\mathbf{Cay})$ is invariant under the action of F_4.*

The proof is obvious from (3.13) and the fact that the trace function is invariant under automorphisms. Now, we are going to show that the inclusion

$$\varphi : P_2(\mathbf{Cay}) \to H(3, \mathbf{Cay}) \tag{3.17}$$

is a minimum imbedding.

As in § 2, we first consider the height functions h_A, $A \in H(3, \mathbf{Cay})$. We can also treat $x = (x_1, x_2, x_3) \in \mathbf{Cay}^3$, $x^*x = 1$, as a sort of homogeneous coordinate for $P_2(\mathbf{Cay})$. Owing to Lemmas 1, 3 and 4, we may assume that

$$A = \lambda_1 E_1 + \lambda_2 E_2 + \lambda_3 E_3 \; . \tag{3.18}$$

Then

$$h_A(x) = \lambda_1 n(x_1) + \lambda_2 n(x_2) + \lambda_3 n(x_3) \; . \tag{3.19}$$

$(x_1, x_2, x_3) = (y_1, y_2, y_3)$ implies $n(x_i) = n(y_i)$, $i = 1, 2, 3$. Hence it makes sense to consider the following local coordinate system. Let U_1 be the set of $x = (x_1, x_2, x_3)$ with $x_1 \neq 0$, and let

$$|x_1| x_1^{-1} x_i = u_{i0} + u_{i1} j_1 + \cdots + u_{i7} j_7 \tag{3.20}$$

where $|x_1| = n(x_1)^{1/2}$. Then

$$u_{i\alpha} : U_1 \to \mathbf{R} \; , \quad 2 \leq i \leq 3 \; , \quad 0 \leq \alpha \leq 7 \; , \tag{3.21}$$

are the required coordinate functions mapping U_1 diffeomorphically onto the open unit ball in $\mathbf{R}^{16}$. Clearly

$$n(x_i) = \sum_{\alpha=0}^{7} u_{i\alpha}^2 \; , \tag{3.22}$$

$$n(x_1) = 1 - n(x_2) - n(x_3) = 1 - \sum_{\substack{2 \leq i \leq 3 \\ 0 \leq \alpha \leq 7}} u_{i\alpha}^2 \; , \tag{3.23}$$

so that

$$h_A(x) = \lambda_1 \sum_{\substack{2 \leq i \leq 3 \\ 0 \leq \alpha \leq 7}} (\lambda_i - \lambda_1) u_{i\alpha}^2 \tag{3.24}$$

throughout the coordinate neighborhood U_1. Thus the only critical point of h_A with U_1 lies at the center point

$$P_1 = (1, 0, 0)$$

of the coordinate system. At this point, h_A is nondegenerate if and only if the other two eigenvalues are distinct from λ_1.

Similarly one can consider other coordinate neighborhoods centered at the points

$$P_2 = (0, 1, 0)\ , \qquad P_3 = (0, 0, 1)\ .$$

If follows that P_1, P_2, P_3 are the only critical points of h_A. Thus we have

Theorem 3.5. *For $A \in H(3, \boldsymbol{Cay})$ the height function h_A defined over $P_2(\boldsymbol{Cay})$ is nondegenerate and has exactly three isolated critical points if and only if all three eigenvalues are distinct from each other.*

Remark 3.6. If h_A is nondegenerate, the indices at three different critical points are respectively 0, 8, 16. From the cell decomposition of h_A, it follows that the sum of Betti numbers $b(P_2(\boldsymbol{Cay})) = 3$. Therefore every height function has the minimum number of critical points. Hence

Theorem 3.7. *The inclusion*

$$\varphi : P_2(\boldsymbol{Cay}) \to H_1(3, \boldsymbol{Cay})$$

is a substantial, minimal, isometric and equivariant imbedding.

The equivariance follows from the fact that φ is an inclusion. φ is isometric, since every equivariant imbedding of an irreducible symmetric space is isometric (cf. Theorem 2.5).

4. Appendix

This appendix is based on Chevalley's *Theory of Lie Groups* [2, Chapter I, §§ III-VII]. Most proofs are omitted, which can be either found, or proved by similar arguments, for the complex case in that book.

Lemma 4.1. *For each $A \in M(n + 1, Q)$, there exists an $x \in Q^{n+1}$ such that $Ax = x\lambda$, $\lambda \in Q$.*

The author is indebted to Professor Kobayashi for the following simple proof.

Proof. If A is singular, then there is an $x \in Q^{n+1}$ such that $x \neq 0$ and $Ax = 0$. Suppose A is nonsingular. Since $GL(n + 1, Q)$ is connected, $A \sim I : Q^{n+1} \to Q^{n+1}$, where I is the identity transformation. Let A' be the induced map on $P_n(Q)$. Then $A' \sim Id : P_n(Q) \to P_n(Q)$, where Id is the identity map of $P_n(Q)$. Therefore the Lefschetz number

$$L(A') = L(Id) = \chi(P_n(Q)) = n + 1 \neq 0. \tag{4.1}$$

Hence, by Lefschetz fixed point theorem, A' has a fixed point. Equivalently, there exists an $x \in Q^{n+1}$ such that $Ax = \lambda x$.

Lemma 4.2 [2, p. 21, Proposition 2]. *If a is a unit vector in Q^{n+1}, there exists a symplectic matrix X such that $Xe_1 = a$, where $e_1 = (1, 0, \cdots, 0)$.*

Theorem 4.3. *For each* $A \in H(n+1, Q)$, *there exists an* $X \in Sp(n+1)$ *such that* XAX^* *is a diagonal matrix.*

The existence of an eigenvector (Lemma 4.1) and Lemma 4.2 make the inductive process possible. The proof is almost the same as the complex case [2, pp. 12-13]. We have also

Theorem 4.4. *The eigenvalues of a Hermitian quaternionic matrix are real numbers.*

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